4

Determinants

Structure

4.1. Introduction. In this chapter, we shall learn to evaluate the determinant of a square matrix and then with the help of this we will solve some system of linear equations having two or three variables.

4.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Determinants.
(ii) Inverse of a matrix.
(iii) Applying row and column operations wherever required.
(iv) Solving system linear equations.

4.1.2. Keywords. Matrix, Determinant, Inverse of a Matrix, Adjoint of a matrix.

4.2. Determinants.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be a square matrix of order n. Then a unique number can be associated to A, known as its determinant. The determinant of A can be denoted by:

$$\det A \quad \text{or} \quad |A| \quad \text{or} \quad |a_{ij}| \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

1. If $A = (a_{11})_{1\times 1}$, then the determinant of A is defined as $|A| = a_{11}$.
2. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$, then determinant of A is defined as $|A| = \begin{vmatrix} a_{11} & & a_{12} \\ & \searrow \ \swarrow & \\ a_{21} & & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.
3. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

4. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, then determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.$$

For matrices of higher order similar procedure can be adopted.

4.2.1. Singular and Non-singular Matrices:

Any square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

4.2.2. Minors and Cofactors.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ be any matrix, then minor of an element a_{ij}, denoted by M_{ij} is the determinant of elements of A obtained by removing ith row and jth column of A, keeping the order of rest rows and columns unchanged.

Thus, $M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix}$.

The cofactor of a_{ij}, denoted by A_{ij}, is defined to be $(-1)^{i+j}\ M_{ij}$.

For example, let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a square matrix of order 2. Then, minors are obtained as

$$M_{11} = \text{Minor of } a_{11} = a_{22}, \quad M_{12} = \text{Minor of } a_{12} = a_{21}$$

$$M_{21} = \text{Minor of } a_{21} = a_{12}, \quad M_{22} = \text{Minor of } a_{22} = a_{11}$$

and cofactors are obtained by

$$A_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} . M_{11} = a_{22},$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^{1+2} . M_{12} = -a_{21},$$

$$A_{21} = \text{Cofactor of } a_{21} = (-1)^{2+1} . M_{21} = -a_{12},$$

$$A_{22} = \text{Cofactor of } a_{22} = (-1)^{2+2} . M_{22} = a_{11}.$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3. Then,

$$M_{11} = \text{Minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$$

$$M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}.$$

Minors for remaining elements can be obtained in the similar pattern. Further,

$$A_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} M_{11} = M_{11} = (a_{22}a_{33} - a_{23}a_{32})$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$A_{13} = \text{Cofactor of } a_{13} = (-1)^{1+2} M_{13} = M_{13} = a_{21}a_{32} - a_{22}a_{31}.$$

Cofactors for remaining elements can be obtained in the similar pattern.

Remark. It should be noted that if A is any matrix, then its determinant is the sum of products of elements of any row and their corresponding cofactors. Thus,

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} .$$

4.2.3. Example. Solve for x:

$$\begin{vmatrix} 5x & 3 \\ 2 & 3 \end{vmatrix} = 5.$$

Solution. Here, $\begin{vmatrix} 5x & 3 \\ 2 & 3 \end{vmatrix} = 5 \quad \Rightarrow 15x - 6 = 5 \Rightarrow 15x = 11 \Rightarrow x = \frac{11}{15}$.

4.2.4. Example. Write the minors and co-factors of all the elements in $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 0 & 2 \\ 5 & 1 & 3 \end{pmatrix}$.

Solution. Let M_{ij} and A_{ij} denotes the minor and co-factor of the element a_{ij} respectively, then

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 0 - 2 = -2 \text{ and } A_{11} = (-1)^{1+1} M_{11} = -2 .$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} = 9 - 10 = -1 \text{ and } A_{12} = (-1)^{1+2} M_{12} = 1 .$$

$$M_{13} = \text{minor of } a_{13} = \begin{vmatrix} 3 & 0 \\ 5 & 1 \end{vmatrix} = 3 - 0 = 3 \text{ and } A_{13} = (-1)^{1+3} M_{13} = 3 .$$

$$M_{21} = \text{minor of } a_{21} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 0 - 2 = -2 \text{ and } A_{21} = (-1)^{2+1} M_{21} = 2 .$$

$$M_{22} = \text{minor of } a_{22} = \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} = 3 - 10 = -7 \text{ and } A_{22} = (-1)^{2+2} M_{22} = -7 .$$

$$M_{23} = \text{minor of } a_{23} = \begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix} = 1 - 0 = 1 \text{ and } A_{23} = (-1)^{2+3} M_{23} = -1 .$$

$$M_{31} = \text{minor of } a_{31} = \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0 - 0 = 0 \text{ and } A_{31} = (-1)^{3+1} M_{31} = 0 .$$

$$M_{32} = \text{minor of } a_{32} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = 2 - 6 = -4 \text{ and } A_{32} = (-1)^{3+2} M_{32} = 4 .$$

$$M_{33} = \text{minor of } a_{33} = \begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0 - 0 = 0 \text{ and } A_{33} = (-1)^{3+3} M_{33} = 0 .$$

4.2.5. Example. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ find $|A|$ by expanding along first row and second column and verify that the value is same.

Solution. Expanding by first row, we have

$$|A| = 1(-1)^{1+1}\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1(-3) \; -2(-6) + 3(-3) = 0.$$

Again, expanding by second column, we have

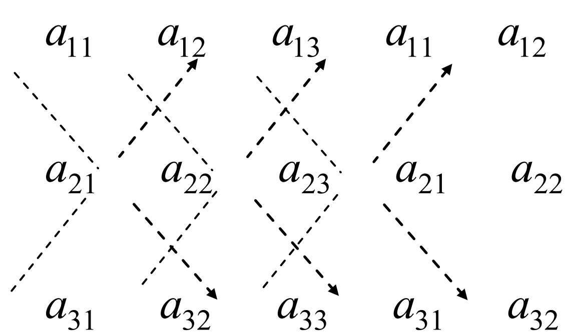
$$|A| = 2(-1)^{2+1}\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^{2+3}\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -2(36-42)+5(9-21)-8(6-12) = 0.$$

Thus the determinant obtained by expanding along different rows are same.

4.2.6. Determinant using Sarrus Method.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

First write five columns in the following order:



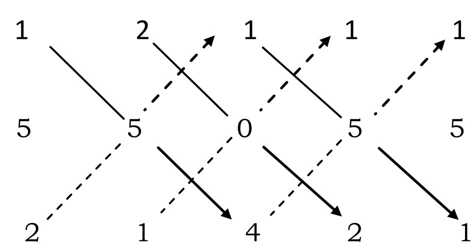
The value of $|A|$ is given by adding the products of the diagonals going from top to bottom and subtracting the products of the diagonals going from bottom to top. Thus

$$|A| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Note. Sarrus Method is used only for determinant of order 2 and 3.

4.2.7. Example. Evaluate the determinant $\begin{vmatrix} 1 & 2 & 1 \\ 5 & 5 & 0 \\ 2 & 1 & 4 \end{vmatrix}$ using Sarrus Method.

Solution. By Sarrus diagram,



we have, $|A| = (1.5.4 + 2.0.2 + 1.5.1) - (2.5.1 + 1.0.1 + 4.5.1)$

$= 25 - 30 = -5.$

4.2.8. Exercise.

1. Which of the following matrices are singular and which are non-singular.

(i) $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 5 \\ 0 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 2 \\ 4 & 3 & 7 \end{bmatrix}$

2. For what value of λ, the matrix $\begin{bmatrix} 7 & 1 \\ 2 & \lambda \end{bmatrix}$ is singular.

3. Find the minors and cofactors the following matrices:

(i) $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 1 \\ 2 & 3 \end{bmatrix}$

4. Solve the following equations for x:

(i) $\begin{vmatrix} 3x & 4 \\ 0 & 2 \end{vmatrix} = 8$ (ii) $\begin{vmatrix} x & x \\ -5 & x \end{vmatrix} = -6$

5. Find the following determinants.

(i) $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix}$ (iii) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

6. Find the determinant using Sarrus Method: $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 3 & 1 \\ 2 & 4 & 1 \end{vmatrix}$

Answer.

1. (i) Singular. (ii) *N*on-singular. (iii) *N*on-singular.

2. $\lambda = \frac{2}{7}$.

3. (i) $M_{11} = -1,\ M_{12} = 2,\ M_{21} = -1\ ,\ M_{22} = 1\ ,\ A_{11} = -1,\ A_{12} = -2,\ A_{21} = 1,\ A_{22} = 1$

(ii) $M_{11} = 3,\ M_{12} = 2,\ M_{21} = 1\ ,\ M_{22} = 7\ ,\ A_{11} = 3,\ A_{12} = -2,\ A_{21} = -1,\ A_{22} = 7$

4. (i) $\frac{4}{3}$ (ii) $x = -3\ , -2$

5. (i) – 7 (ii) – 9 (iii) 4*abc*

6. – 9

4.3. Properties of Determinants.

Using the following properties of determinants, we can evaluate the determinant of a matrix without using the evaluation methods discussed earlier.

We will use the notations $R_1, R_2, ..., C_1, C_2, ...$ to denote row one, row two, …, column one, column two, ... etc. of a matrix.

1. The value of determinant remains unchanged if rows (columns) are changed into columns (rows), that is, if A is a matrix, then $|A| = |A'|$.
2. If two adjacent rows (columns) of a determinant are interchanged then the value of determinant is multiplied by -1.
3. If any two rows (columns) are identical then the value of the determinant is zero.
4. If any two rows (columns) are multiples of each other then the determinant is zero.
5. If all entries of any row (column) are zero then the determinant is zero.
6. If each element in a row (column) of a determinant is multiplied by any scalar then the determinant is also multiplied by same scalar.
7. If every element of any row (column) is the sum (or difference) of two or more quantities, then the determinant can also be expressed as the sum (difference) of two or more determinants of same order.

   For example, let $\Delta = \begin{vmatrix} 7 & 2 & 1 \\ 4 & 5 & 2 \\ 3 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5+2 & 2 & 1 \\ 3+1 & 5 & 2 \\ 2+1 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 5 & 2 \\ 2 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 & 1 \\ 1 & 5 & 2 \\ 1 & 3 & 2 \end{vmatrix}$

8. If to every element of a row (column) of a determinant be added or subtracted equal multiples of the corresponding elements of one or more rows (or columns) then the value of the determinant unchanged.
9. The determinant of product of square matrices of same order is equal to the product of the determinants of matrices, that is, $|AB| = |A|.|B|$

4.3.1. Example. Without expanding show that following determinant vanishes.

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 1 & 1 & 8 \end{vmatrix}$ **(ii)** $\begin{vmatrix} 29 & 1 & 4 \\ 33 & 5 & 4 \\ 17 & 3 & 2 \end{vmatrix}$

Solution. (i) Let $\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 1 & 1 & 8 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and using property 5,we get

$$\Delta = \begin{vmatrix} 1 & 3 & 5 \\ 2-2 & 6-6 & 10-10 \\ 1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \\ 1 & 1 & 8 \end{vmatrix} = 0$$

(ii) Let
$$\Delta = \begin{vmatrix} 29 & 1 & 4 \\ 33 & 5 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 7C_3$ and using property 3, we get

$$\Delta = \begin{vmatrix} 29-28 & 1 & 4 \\ 33-28 & 5 & 4 \\ 17-14 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 4 \\ 5 & 5 & 4 \\ 3 & 3 & 2 \end{vmatrix} = 0$$

4.3.2. Example. Using properties of determinants, show that $\begin{vmatrix} \mathbf{1} & \boldsymbol{a} & \boldsymbol{a^2 - bc} \\ \mathbf{1} & \boldsymbol{b} & \boldsymbol{b^2 - ca} \\ \mathbf{1} & \boldsymbol{c} & \boldsymbol{c^2 - ab} \end{vmatrix} = \mathbf{0}$.

Solution : Let $\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$ then

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Multiplying R_1,R_2 and R_3 of second term of Δ by a, b and c, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Applying $C_1 \leftrightarrow C_2$ in second term of Δ, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

4.3.3. Example. Show that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution : Let

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + 0$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 2a & 6a & 3c \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2 c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \,.\, 0 + a^2 c \,.\, 0$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} = a^3 \times 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = a^3(7-6) = a^3$$

4.3.4. Example. Evaluate $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$.

Solution: Let $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0$$

as first and third rows are identical.

4.3.5. Example. Show that $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$.

Solution. Let $\Delta = \begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by a, b, and c respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} (b+c)^2 a & ba^2 & ca^2 \\ ab^2 & (c+a)^2 b & cb^2 \\ ac^2 & bc^2 & (a+b)^2 c \end{vmatrix}$$

Taking a, b and c common from C_1,C_2 and C_3, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

Taking $a+b+c$ common from C_1 and C_2, we get

$$\Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Applying $C_1 \to C_1(a)$, $C_2 \to C_2(b)$

$$\Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3,\ C_2 \to C_2 + C_3$

$$\Delta = \frac{(a+b+c)^2}{ab}\begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

Taking a, b and $2ab$ common from R_1, R_2 and R_3 respectively

$$\Delta = \frac{(a+b+c)^2}{ab}\ ab\ .\ 2ab\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix}$$

Now expanding along R_3, we get

$$\Delta = 2ab(a+b+c)^2\begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} = 2ab(a+b+c)^2\ [(b+c)(c+a)-ab) = 2abc(a+b+c)^3$$

4.3.6. Example. Show that $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$.

Solution : Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

4.3.7. Exercise.

1. Without expanding show that following determinant vanishes.

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$ (ii) $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ (iii) $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ (iv) $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ac \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$

(v) $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & a & abc \\ 1 & b & abc \\ 1 & c & abc \end{vmatrix}$ (vii) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ (viii) $\begin{vmatrix} 1 & \text{a} & \text{abc} \\ 1 & \text{b} & \text{abc} \\ 1 & \text{c} & \text{abc} \end{vmatrix}$

2. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+bc+ca-a^2-b^2-c^2)$

3. Show that

(i) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$ (ii) $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

(iii) $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$ (iv) $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (b-a)(c-a)(c-b)$

(v) $\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$

4. Show that

(i) $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$ (ii) $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$

5. Show that

(i) $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$ (ii) $\begin{vmatrix} b+c & a & b \\ c+a & c & a \\ a+b & b & c \end{vmatrix} = (a+b+c)(a-c)^2$

4.4. Adjoint of a Matrix.

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then the adjoint of matrix A is defined as

$$\text{adj}A = [A_{ij}]'$$

where A_{ij} is the corresponding co-factor of a_{ij}.

4.4.1. Example. Find the adjoint of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution. Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

By definitions of Cofactors:

$$A_{11} = \text{cofactor of } a_{11} = 4$$

$$A_{12} = \text{cofactor of } a_{12} = -3$$

$$A_{21} = \text{cofactor of } a_{21} = -2$$

$$A_{22} = \text{cofactor of } a_{22} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

4.4.2. Theorem. If A is square matrix of order $n \times n$, then prove that

$$A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)A.$$

4.4.3. Example. Find adjoint of $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and also verify that $(\text{adj}A)A = |A|\,I_2 = A(\text{adj}\,A)$.

Solution : Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

Cofactors of elements of A are:

$A_{11} = \text{cofactor of } a_{11} = 5$, $A_{12} = \text{cofactor of } a_{12} = -3$

$A_{21} = \text{cofactor of } a_{21} = -2$, $A_{22} = \text{cofactor of } a_{22} = 1$

Thus, $\text{adj}A = \begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}' = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1$

So $A(\text{adj}A) = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = |A|\,I_2$

Again $(\text{adj}AA) = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = |A|\,I_2$

So, we get

$$A(\text{adj } A) = |A|\,I_2 = (\text{adj } A)\,A$$

4.4.4. Exercise.

1. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ verify, adj(AB) = (adj$B$) (adj$A$).

2. Find the adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$. Also show that $A.(\text{adj } A) = |A|.I_3 = (\text{adj } A).A$.

3. Find the adjoint of following matrices.

   (i) $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

4.5. Inverse of a Matrix.

A square matrix of order n is invertible if there exist a square matrix B of same order such that $AB = I_n = BA$.

In such a case, we say that inverse of A is B and inverse of B is A and we write

$$A^{-1} = B, B^{-1} = A .$$

If inverse of a matrix exists, then it is called an invertible matrix.

4.5.1. Theorem. A square matrix is invertible iff it is non-singular.

Proof. Let A be an invertible matrix. Then, there exists a matrix B such that

$$AB = I_n = BA$$

$\Rightarrow$ $|AB| = |I_n|$

$\Rightarrow$ $|A||B| = 1$

$\Rightarrow$ $|A| \neq 0$

$\Rightarrow$ A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n that is, $|A| \neq 0$. Then, we know that

$$A(\text{adj } A) = |A| I_n = (\text{adj } A)A$$

Dividing both sides by $|A|$,

$$\Rightarrow \quad A\left(\frac{1}{|A|}\text{adj } A\right) = I_n = \left(\frac{1}{|A|}\text{adj } A\right)A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\text{adj } A$$

Hence, A is an invertible matrix.

Remark. Due to the above theorem, we can say that the inverse of a non-singular matrix A is given by

$$A^{-1} = \frac{adj\ A}{|A|}$$

4.5.2. Theorem. If A is an invertible square matrix, then A' is also invertible and

$$(A')^{-1} = \left(A^{-1}\right)'.$$

Proof. Since A is an invertible matrix, so $|A| \neq 0$, and thus $|A'| \neq 0$, which implies A' is also invertible.

Now, $$AA^{-1} = I_n = A^{-1}A$$

$$\Rightarrow \quad \left(AA^{-1}\right)' = (I_n) = \left(A^{-1}A\right)'$$

$$\Rightarrow \quad \left(A^{-1}\right)'(A') = I_n = A'\left(A^{-1}\right)'$$

$$\Rightarrow \quad (A')^{-1} = \left(A^{-1}\right)'$$

4.5.3. Theorem. If A and B are invertible matrices of the same order, then so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. It is given that A and B are invertible matrices, therefore $|A| \neq 0$ and $|B| \neq 0$

$$\Rightarrow \quad |A||B| \neq 0$$

$$\Rightarrow \quad |AB| \neq 0$$

$\Rightarrow$ AB is a invertible matrix.

Now,

$$(AB)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n$$

and,

$$\left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}(I_nB) = B^{-1}B = I_n$$

Thus, $$(AB)\left(B^{-1}A^{-1}\right) = I_n = \left(B^{-1}A^{-1}\right)(AB)$$

Hence, $$(AB)^{-1} = B^{-1}A^{-1}.$$

4.5.4. Theorem. Inverse of an invertible matrix is always unique.

Proof. Let A be an invertible matrix of order $n \times n$ having matrices B and C as its two inverses. Then,

$$AB = BA = I_n \text{ and } AC = CA = I_n$$

Now, $AB = I_n$ $\Rightarrow$ $C(AB) = CI_n$

$$\Rightarrow \quad (CA)B = CI_n$$

$$\Rightarrow \quad I_nB = CI_n$$

$$\Rightarrow \quad B = C$$

Hence, inverse of a matrix is unique.

4.5.5. Corollary. If A is an invertible matrix, then $(A^{-1})^{-1} = A$.

Proof. We have,

$$AA^{-1} = I = A^{-1}A$$

$\Rightarrow$ A is the inverse of A^{-1}, that is, $A = (A^{-1})^{-1}$.

4.5.6. Example. Find the inverse of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Solution : Given that $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Therefore, $|A| = 1 + 1 = 2 \neq 0$, which implies A^{-1} exists.

Now, by definition

$$A_{11} = \text{cofactor of } a_{11} = 1$$
$$A_{12} = \text{cofactor of } a_{12} = 1$$
$$A_{21} = \text{cofactor of } a_{21} = -1$$
$$A_{22} = \text{cofactor of } a_{22} = 1$$

Thus, $\text{adj } A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Now $A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

4.5.7. Example. If $A = \begin{bmatrix} \mathbf{2} & \mathbf{7} \\ \mathbf{1} & \mathbf{4} \end{bmatrix}$, show that $A^2 - 6A + I = O$. Hence find A^{-1}.

Solution. Here, $A^2 = A.A = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix}$

So $A^2 - 6A + I = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix} - 6\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 42 \\ 6 & 23 \end{bmatrix} - \begin{bmatrix} 12 & 42 \\ 6 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence, $A^2 - 6A + I = O$.

Now using this we have to find A^{-1}.

$$A^2 - 6A + I = O \Rightarrow \qquad 6A - A^2 = I$$

Now pre-multiplying both sides by A^{-1} we have,

$$A^{-1} = 6I - A$$

So, $A^{-1} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$.

4.5.8. Exercise.

1. Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify your answer.

2. For the matrices $A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$ and show that

$$aA^{-1}=(a^2+bc+1)I_2-aA.$$

4. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2-4A-5I = O$ and hence find A^{-1}.

4.6. Inverse of a Matrix by using Elementary Operations.

4.6.1. Elementary Operations. To obtain inverse of a matrix sometimes we use some operations on a given matrix called elementary operations.

These are of two types:

1. **Elementary row operations.** Elementary operation on rows of a matrix are known as elementary row operation. Following are the various types of elementary row operations

i) The interchange of any two rows. By $R_i \leftrightarrow R_j$, we mean interchanging *i*th row of the given matrix with *j*th row.

ii) The multiplication of the elements of row by a non-zero number. By $R_i \to k\, R_i$, we mean that the elements of *i*th row of the given matrix are multiplied by *k*.

iii) Adding to the elements of a row, the corresponding elements of any other row multiplied by any scalar *k*. By $R_i \to R_i + kR_j$, we mean that the elements of *j*th row of the given matrix are multiplied by *k* and then the elements are added to corresponding elements of *i*th row.

Remark. An elementary row operation on the product of two matrices is equivalent to the same elementary row operation on the pre-factor.

4.6.2. To find inverse of a square matrix by using elementary row operation.

Let A be a non-singular matrix. So, it can be written as $A = IA$, where I is identity matrix. Now apply elementary row operations on A to convert it to I and on right side apply these operations as applied on left side to I. If I is converted to B, then this matrix B is inverse of A.

2. **Elementary column Operations.** The similar operations are defined for columns and known as elementary column operations. Also to find inverse of a matrix A this time we will consider $A = AI$ and then apply elementary columns operations on A to convert it to I and on right side apply these operations as applied on left side to I. If I is converted to B, then this matrix B is inverse of A.

4.6.3. Example. Find inverse using elementary row operations of $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$.

Solution : Given that $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, then $|A| = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 \neq 0$. So A^{-1} exists.

Now let $A = IA$, which implies $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} A$$

Therefore, $A^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$.

4.6.4. Example. Find the inverse of matrix $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ using elementary column operation.

Solution. Clearly A is invertible.

Now let $A = AI$, which implies $\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Applying $C_2 \to C_2 - 3C_1$, we get

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$.

4.6.5. Exercise.

1. Find the inverse of matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ by using elementary row operations.
2. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ using elementary row operations.

3. Using elementary column operations, find the inverse of matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{-7} & \mathbf{2} & \mathbf{1} \end{bmatrix}$.

4. Find the inverse of $A = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{-2} & \mathbf{4} \end{bmatrix}$ by using elementary column operations.

4.7. Solution of Simultaneous Linear Equations.

A system of linear equation has either unique solution or infinitely many solutions or no solution. If a system of linear equations has a solution (whether unique or infinite), then the system is said to be **consistent** and if the system has no solution, it is said to be **inconsistent**.

4.7.1. Cramer's Rule to Solve the Linear Equations.

1. System of Linear Equation of two variables *x* and *y*.

First we consider a system of linear equations in two variables x and y:

$$ax + by = d_1$$
$$cx + dy = d_2$$

We define D as the determinant obtained from the coefficients of x and y, D_1 and D_2 are determinants obtained by replacing first and second column respectively of D by $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$. Thus,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix},\ D_1 = \begin{vmatrix} d_1 & b \\ d_2 & d \end{vmatrix},\ D_2 = \begin{vmatrix} a & d_1 \\ c & d_2 \end{vmatrix}$$

If $D \neq 0$, then the system has a unique solution given by

$$x = \frac{D_1}{D},\quad y = \frac{D_2}{D}.$$

2. System of Linear Equation of two variables *x, y* and *z*.

Now we consider a system of linear equations in three variables x and y and z:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Then as defined in case of two variables, we define the following:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},\quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},\quad D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix},\quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $D \neq 0$, then the system has unique solution and given by

$$x = \frac{D_1}{D},\quad y = \frac{D_2}{D},\quad z = \frac{D_3}{D}$$

Remark. If $D = 0$, then the system has either infinitely many solutions or no solution. However, the systems with such solutions are not included in the syllabi.

4.7.2. Example. Solve the following system of equations using Cramer's Rule

$$x+y=5$$
$$x+2y=15$$

Solution. Given system of equations is

$$x+y=5$$
$$x+2y=15$$

Then, by definition

$$D=\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix}=2-1=1\neq 0$$

Therefore, the system has a unique solution.

Now $$D_1=\begin{vmatrix}5 & 1\\ 15 & 2\end{vmatrix}=10-15=-5$$

and $$D_2=\begin{vmatrix}1 & 5\\ 1 & 15\end{vmatrix}=15-5=10.$$

Then, by Cramer's Rule, the unique solution is given by

$$x=\frac{D_1}{D}=-\frac{5}{1}=-5, \qquad y=\frac{D_2}{D}=\frac{10}{1}=10.$$

So, $x=-35$, $y=25$ is a solution.

4.7.3. Exercise. Solve the following system of equations by using Cramer's Rule:

1. $\begin{aligned} \mathbf{x+y+z} &= \mathbf{1}\\ \mathbf{3x+5y+6z} &= \mathbf{4}\\ \mathbf{9x+2y-36z} &= \mathbf{17}\end{aligned}$

2. $\begin{aligned} 2y-3z &= 0\\ x+3y &= -4\\ 3x+4y &= 3\end{aligned}$

3. $\begin{aligned} 2x+3y &= 7\\ 4x-5y &= 3\end{aligned}$

4. The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to it, we get 7. By adding second and third numbers to three times the first number, we get 12. Find the numbers.

5. The perimeter of a triangle is 45 cm. The longest side exceeds the shortest side by 8 cm and sum of the length of the longest and the shortest side is twice the length of the other side. Find the lengths of sides of the triangle.

6. Find a, b, c when $f(x)=ax^2+bx+c$, $f(1)=1$, $f(2)=2$, $f(0)=4$. Determine the quadratic function $f(x)$ and find its value when $x=0$.

Answers.

1. $x=\frac{1}{3}$, $y=1$, $z=-\frac{1}{3}$
2. $x=5, y=-3, z=-2$
3. $x=2$, $y=1$
4. 3,1,2
5. 19 cm, 15 cm, 11 cm
6. $2x^2-5x+4$, 4

4.7.4. Matrix Method to solve system of linear equations.

1. System of Linear Equation of two variables *x* and *y*.

First we consider a system of linear equations in two variables *x* and *y*:

$$a_1 x + b_1 y = d_1$$

$$a_2 x + b_2 y = d_2$$

We define $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Then the given system of equations can be written in matrix form as

$$AX=B.$$

If $|A| \neq 0$, then the system has unique solution given by

$$X = A^{-1} B.$$

2. System of Linear Equation of three variables *x, y* and *z*.

First we consider a system of linear equations in two variables *x, y* and *z*:

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Define $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

If $|A| \neq 0$, then the system has unique solution given by

$$X = A^{-1} B$$

Remark. If $|A| = 0$, then the system has either infinitely many solutions or no solution. However, the systems with such solutions are not included in the syllabi.

4.7.5. Example. Solve the following system of equations by matrix method:

$$x + y = 1$$
$$2x + y = 2$$

Solution. The given system of equations can be represented in matrix form as $AX = B$ where

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now, $|A| = 1 - 2 = -1 \neq 0$.

Thus, the system has a unique solution given by

$$X = A^{-1} B$$

We need to obtain the inverse of A, for this cofactors of elements of A are

$$A_{11} = 1 \quad , \quad A_{12} = -2 \quad , \quad A_{21} = -1 \quad , \quad A_{22} = 1.$$

Thus, $\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$

and $A^{-1} = \frac{1}{|A|} \text{adj } A = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$

Therefore, the solution can be obtained from

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence $x = 2$, $y = -1$ is a solution.

4.7.6. Exercise. Solve the following system of equations:

1. $$\begin{aligned} 2x + 8y + 5z &= 6 \\ x + y + z &= -2 \\ x + 2y - z &= 2 \end{aligned}$$

2. $$\begin{aligned} \frac{2}{x} - \frac{3}{y} + \frac{3}{z} &= 10 \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 10 \\ \frac{3}{x} - \frac{1}{y} + \frac{2}{z} &= 13 \end{aligned}$$

Answers.

1. $x = -3, \quad y = 2, \quad z = -1$
2. Use $\frac{1}{x} = u$, $\frac{1}{y} = v$, $\frac{1}{z} = w$, then solving the system we will obtain $u = 2$, $v = 3$, $w = 5$.

4.8. Check Your Progress.

1. Write the minors and cofactors of all elements of $\begin{bmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{bmatrix}$

2. For the matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$. Hence find A^{-1}.

Answers.

1. $M_{11} = -2, \; M_{12} = -7, M_{13} = 3, M_{21} = 5, M_{22} = 7, \; M_{23} = -11, M_{31} = 4, M_{32} = 7, M_{33} = -6$

$A_{11} = -2, \; A_{12} = 7, A_{13} = 3, A_{21} = -5, A_{22} = 7, \; A_{23} = 11, A_{31} = 4, A_{32} = -7, A_{33} = -6$

4.9. Summary. In this chapter, we discussed about determinants of matrices, invertible matrices and the role played by an invertible matrix to solve a system of linear equations having a unique solution.

References.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.